The 2011 Population Census is going to be conducted from June to August. Previous censuses / by-censuses indicated that the population growth in Hong Kong has continued to slow down. Can we predict the population from the past data? Actually, population growth can be modelled by an exponential function. In this issue, we will discuss the exponential function and its applications.

### Applications of Exponential Functions in Daily Life

#### Introduction

An exponential function is a function in the form $y = a^x$, where $a$ is the base and $x$ is the exponent, for $a > 0$ and $a \neq 1$.

For example, $\left(\frac{1}{2}\right)^x$ with base $\frac{1}{2}$, $3^x$ with base 3, $5^{-3x}$ with base 5 and $e^x$ with base $e$. In this issue, we are going to focus on the exponential functions with base $e$ and their applications.

#### What is $e$?

Beside the irrational numbers $\pi$, $\sqrt{2}$, $\sqrt[3]{5}$ that we learnt in junior forms, $e$ is another irrational number. $e$ was discovered after lots of people’s work and is known as Euler’s number, being named after the Swiss mathematician Leonhard Euler (1707 – 1783).

**Definition:**

\[
e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n
\]

\[
= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots
\]

\[
= 2.718281828
\]
Using $e$ as the base, we can express the exponential function $e^x$ as follows:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

We call the infinite series of $e^x$ the exponential series. The exponential series is applied to all values of $x$. We can also expand functions such as $e^{f(x)}$ by using the series.

For example, when $f(x) = -x$,

$$e^{-x} = \lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^n$$

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-x)^n}{n!} + \cdots$$

Applications of Exponential Functions

Exponential functions play an important role in many fields. The function $e^{f(t)}$ can be used to describe phenomena which change over time $t$. Below we will talk about radioactive decay, Newton’s law of cooling and population growth.

Radioactive Decay

In Mathematics Express Issue 13, we introduced the half-life of a radioactive material. Using the radioactive decay formula, the amount of radioactive material remaining after a certain number of half-lives can be found. Indeed, its mass after time $t$ (which is not limited to be a multiple of its half-life) can be modelled by an exponential formula.

If $N_0$ is the amount of a radioactive element at time $t = 0$ and $k (> 0)$ is the decay rate, then the amount $N$ of the element at time $t$ is given by:

$$N = N_0 e^{-kt}$$

The radioactive decay formula can be deduced from the above formula.

When $N = \frac{N_0}{2}$, we have the half-life $b = \frac{\ln 2}{k}$.

Substituting $k = \frac{\ln 2}{b}$ into the above exponential formula, we have $N = N_0 \times \left(\frac{1}{2}\right)^\frac{t}{b}$.

Instead of using the half-life of a radioactive material, we can find the mass by using the decay rate.

Think About It:

A radioactive element decays at a rate of 4.5% per day. It is known that the weight of the element after ten days is 2.55 g. Find the initial weight of the element. (Give the answer correct to the nearest g.)
Newton’s Law of Cooling

When we have a cup of tea left in a room, we may wonder how fast the cup of tea cools. This can be described by Newton’s law of cooling. The temperature \( T \) of the tea after time \( t \) is given by:

\[
T = C + (T_0 - C)e^{-kt},
\]

where \( C \) is the room temperature, \( T_0 \) is the temperature of the tea at time \( t = 0 \) and \( k \) (\( > 0 \)) is the rate of cooling.

\[
f(t) = -kt
\]

Think About It:

Eric warms a cup of tea and then leaves it on a table to cool. The temperature \( T \) (in °F) of the tea after \( t \) hours can be modelled by:

\[
T = 68 + 30e^{-0.13t}
\]

Find the temperature of the tea after one day. (Give the answer correct to the nearest °F.)

Population Growth

The growth in population is an example of exponential growth. Exponential functions can be used to describe phenomena like population of a city or number of bacteria which change over time.

Let \( Q_0 \) be the population at time \( t = 0 \) and \( k \) (\( > 0 \)) be the growth rate. The population at time \( t \) can be found by using the formula:

\[
Q = Q_0e^{kt}
\]

Think About It:

In the 2006 Population By-census, the Hong Kong Resident Population was 6 864 346. Using the average annual growth rate from 2001 – 2006 listed in Table 1, estimate the Hong Kong Resident Population in 2011. (Give the answer correct to the nearest integer.)

<table>
<thead>
<tr>
<th>Period</th>
<th>Average annual growth rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1976 – 1981</td>
<td>3.3</td>
</tr>
<tr>
<td>1981 – 1986</td>
<td>1.5</td>
</tr>
<tr>
<td>1986 – 1991</td>
<td>0.6</td>
</tr>
<tr>
<td>1991 – 1996</td>
<td>1.8</td>
</tr>
<tr>
<td>1996 – 2001</td>
<td>0.9</td>
</tr>
<tr>
<td>2001 – 2006</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Table 1 The average annual growth rate of population in Hong Kong from 1976 to 2006
(Source: Census and Statistics Department)

Conclusion

The number \( e \) is important in Mathematics and its exponent can be used to describe exponential growth and decay. Exponential functions can be applied to various fields such as natural science, commerce and economics. For daily-life phenomena that change over time, we can model them with different exponential functions, which help us study real-life problems by using numerical values.
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